



Time series analysis for coupled neurons

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'Bout me!

1. B.Sc. and M.Sc. in Physics (2015–2020)
2. Ph.D. in Applied Math (2021–2024)
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Dynamical systems (ODEs)

1. ODEs : Ordinary Differential Equations.
2. Rate of change of a physical quantity over time.
3. Generates a data of time series, given an initial time stamp.

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = f(\mathbf{x}, t)$$



Fig. Newton and Leibniz (Wikipedia)



Neurons

1. Neurons are the fundamental units of the nervous system.
2. Billions of neurons couple through 'synapses' to form a cluster of a highly complex neural mass.
3. Their mechanism evolves in time.
4. Thus can be perceived as a 'dynamical system'.

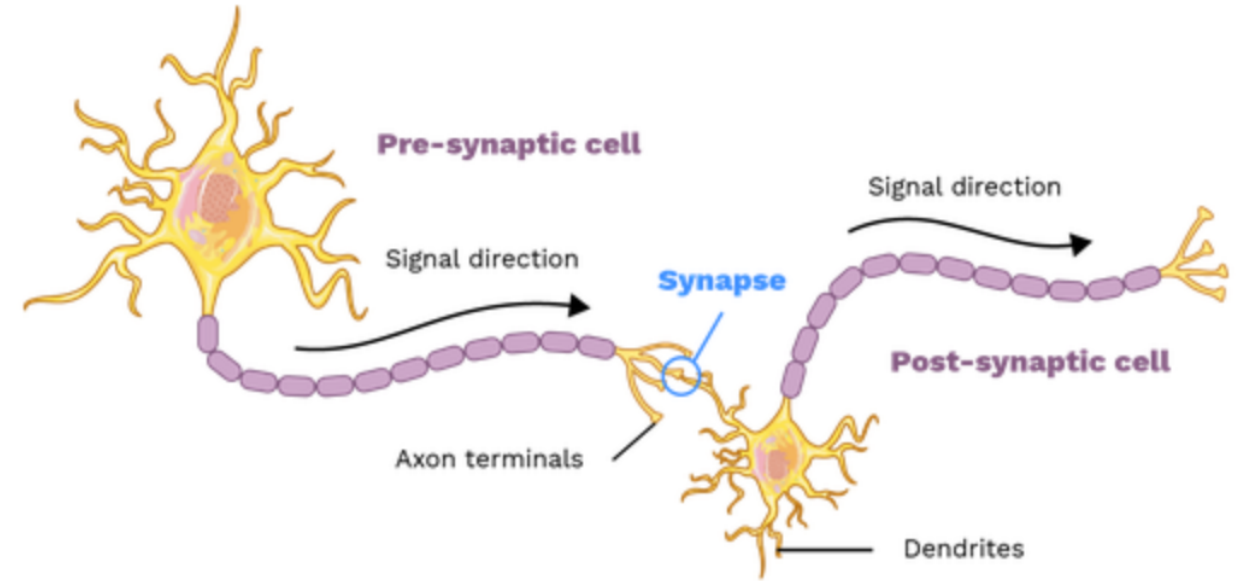


Fig. A typical synapse (theory.labster.com/synapses/)



Chaos

1. In popular term a ‘state of disorder’.
2. In mathematical term, it must be sensitive to initial conditions and have a dense orbit in the phase space.
3. Chaotic systems behave predictably in the beginning before becoming random.

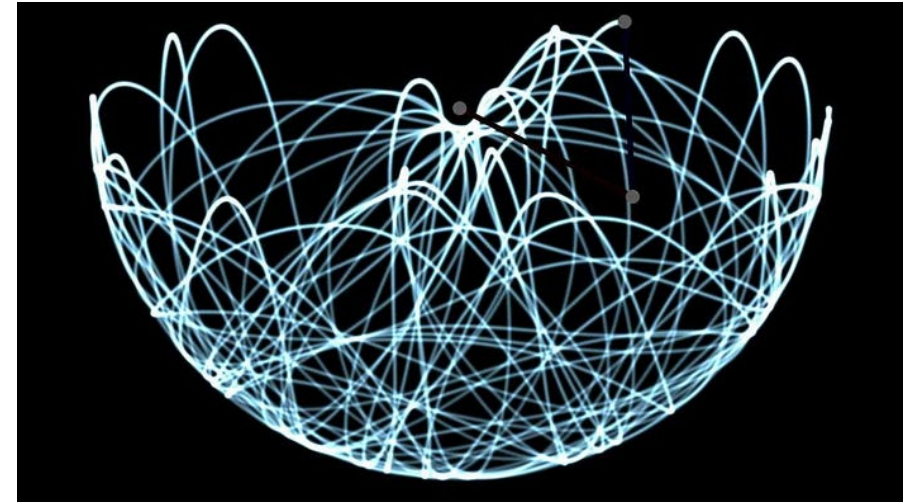
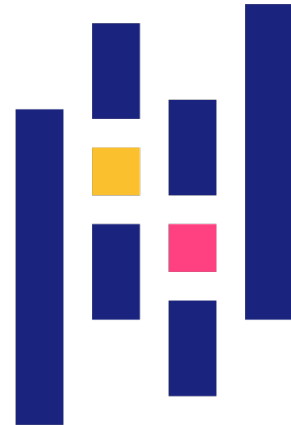


Fig. Double-rod pendulum exhibiting chaos
(Taken from <https://medium.com/@bharatambati/how-the-double-pendulum-creates-simple-chaos-ac49a297fb4d>)



Packages





Single neuron (Schaeffer & Cain, 2018)

1. Simple mathematical model.
2. Parameters selected from empirical experiments.
3. Captures realistic bursting in neurons.
4. Portrays a battery of complex dynamics.
5. Use `solve_ivp()` function from `scipy.integrate` suite to solve initial value problem.

$$\dot{x} = f(x, y, I) = x^2(1 - x) - y + I,$$

$$\dot{y} = g(x, y, I) = Ae^{\alpha x} - \gamma y,$$

$$\dot{I} = h(x, y, I) = \varepsilon \left[\frac{1}{60} \left\{ 1 + \tanh \left(\frac{0.05 - x}{0.001} \right) \right\} - I \right]$$

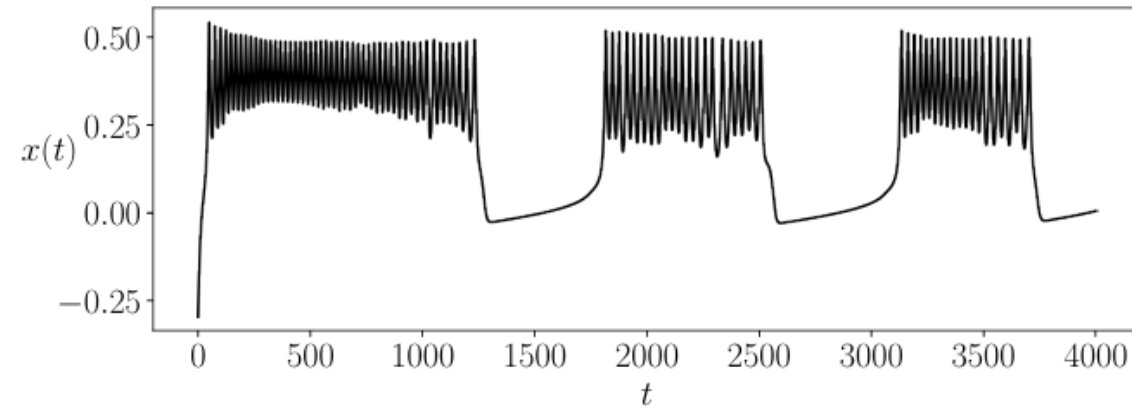
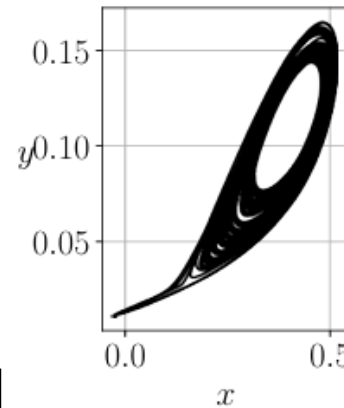


Fig. Phase portrait and time series



“All models are wrong but some are useful”

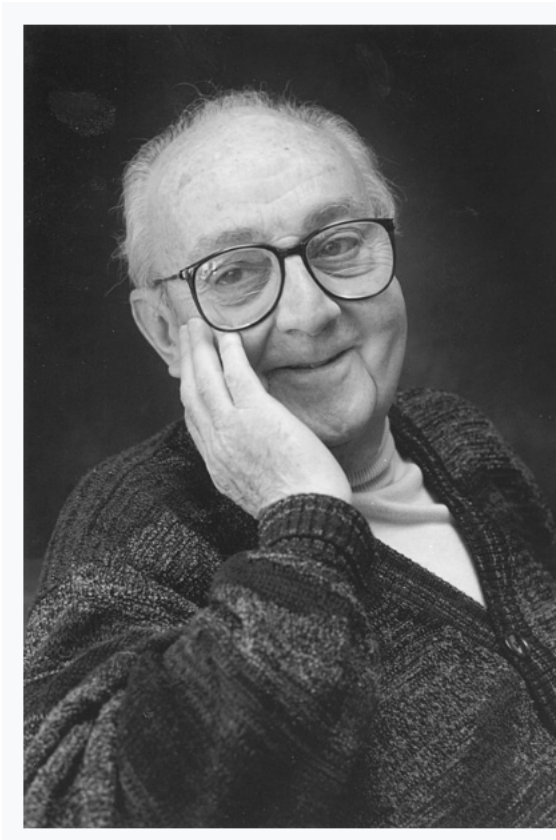


Fig. George Box (Wikipedia)



Simulate a single neuron (code snippet)

```
## Parameters
A = 0.0041
alpha=5.276
gamma = 0.315
epsilon = 0.0005

## Define the function of differential equations
def system(t, vars):
    x1, y1, I1= vars
    dx1dt = x1**2 * (1 - x1) - y1 + I1
    dy1dt = A * np.exp(alpha * x1) - gamma * y1
    dI1dt = epsilon*(1/60*(1+np.tanh((0.05-x1)/0.001))) - I1

    return [dx1dt, dy1dt, dI1dt]

## Initial conditions
x1_0 = np.random.uniform(low=-1, high=1)
y1_0 = 0.1
I1_0 = 0.019

initial_conditions = [x1_0, y1_0, I1_0]
```

```
## Time span for the solution
t_span = (0, 4000)
t_eval = np.linspace(t_span[0], t_span[1], 50000)

## Solve
solution = solve_ivp(system, t_span, initial_conditions, t_eval=t_eval, method='RK45')

## Extract solutions
x1_sol = solution.y[0]
y1_sol = solution.y[1]
I1_sol = solution.y[2]

tt = solution.t
print("done")
```

For the time integration of the differential equations, we use method = 'RK45' which is the explicit Runge-Kutta scheme of order 5(4).



Coupled neurons

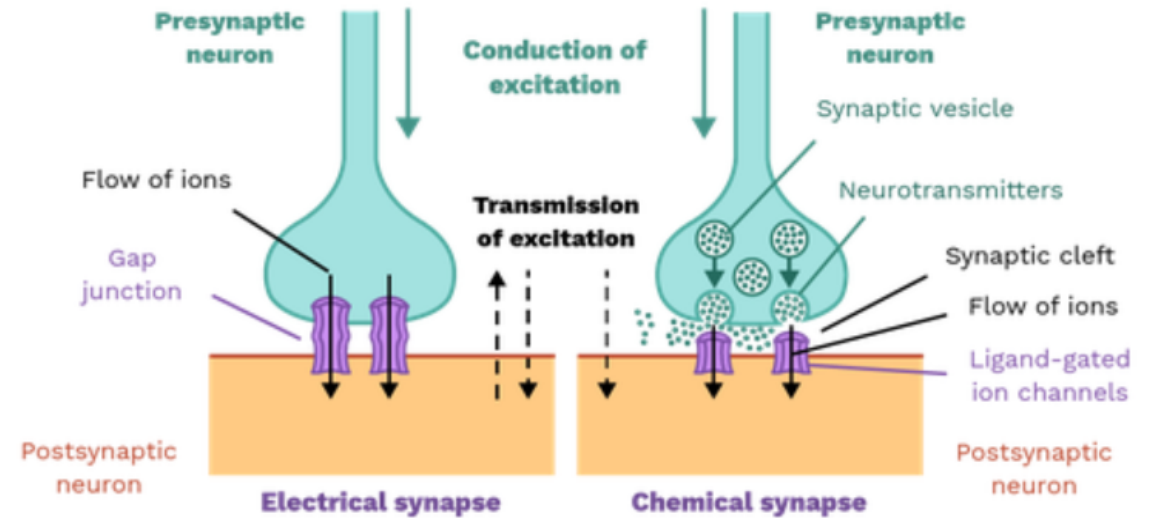
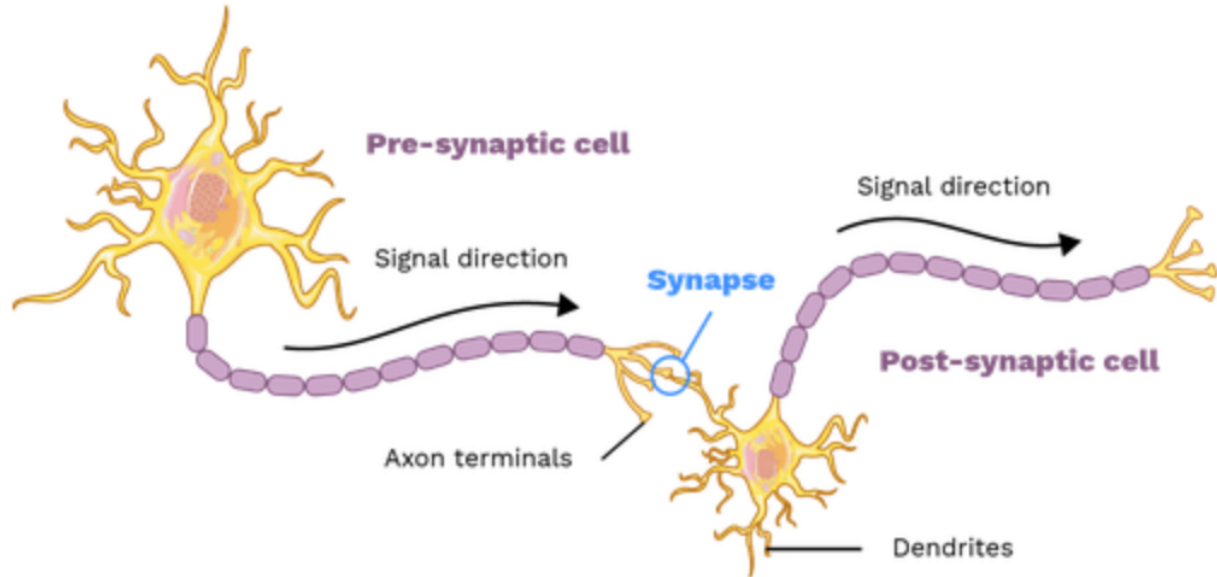


Fig. Coupled neurons (<https://theory.labster.com/synapses/>), and typical electrical and chemical synapses (theory.labster.com/electrical-synapses/)



Toy models of coupled neurons

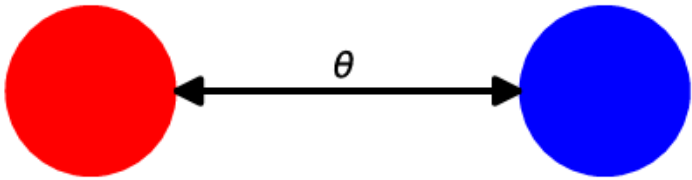


Fig. Electrical (gap-junction) coupling

$$\begin{aligned}\dot{x}_i &= f(x_i, y_i, I_i) + \sum_{j \in B(i)} \theta(x_j - x_i), \\ \dot{y}_i &= g(x_i, y_i, I_i), \\ \dot{I}_i &= h(x_i, y_i, I_i),\end{aligned}$$

Coupling term

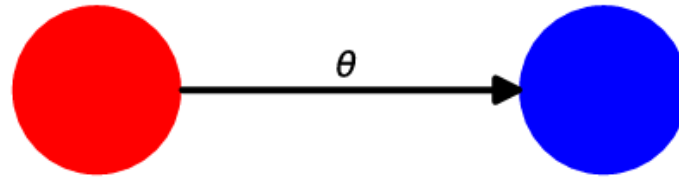


Fig. Chemical coupling

$$\begin{aligned}\dot{x}_1 &= f(x_1, y_1, I_1), \\ \dot{x}_2 &= f(x_2, y_2, I_2) + \theta \frac{v_s - x_2}{1 + \exp\{-\lambda(x_1 - q)\}}, \\ \dot{y}_i &= g(x_i, y_i, I_i), \\ \dot{I}_i &= h(x_i, y_i, I_i),\end{aligned}$$

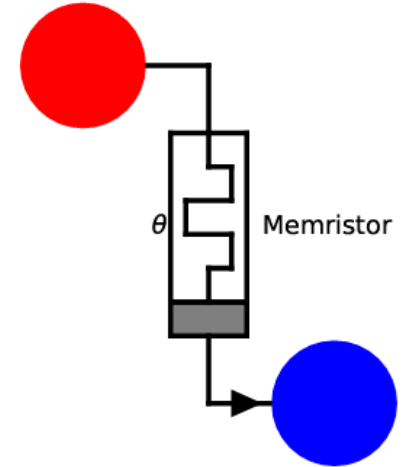


Fig. Electromagnetic coupling

$$\begin{aligned}\dot{x}_1 &= f(x_1, y_1, I_1) + \theta \rho(\phi)(x_2 - x_1), \\ \dot{x}_2 &= f(x_2, y_2, I_2) + \theta \rho(\phi)(x_1 - x_2), \\ \dot{y}_i &= g(x_i, y_i, I_i), \\ \dot{I}_i &= h(x_i, y_i, I_i), \quad i = 1, 2, \\ \dot{\phi} &= \theta(x_1 - x_2)\end{aligned}$$



Simulating coupled neurons

```
def system(t, vars):
    x1, y1, I1, x2, y2, I2 = vars
    dx1dt = x1**2 * (1 - x1) - y1 + I1 + theta*(x2-x1)
    dy1dt = A * np.exp(alpha * x1) - gamma * y1
    dI1dt = epsilon*(1/60*(1+np.tanh((0.05-x1)/0.001))) - I1
    dx2dt = x2**2 * (1 - x2) - y2 + I2 + theta*(x1-x2)
    dy2dt = A * np.exp(alpha * x2) - gamma * y2
    dI2dt = epsilon*(1/60*(1+np.tanh((0.05-x2)/0.001))) - I2

    return [dx1dt, dy1dt, dI1dt, dx2dt, dy2dt, dI2dt]
```

```
def system(t, vars):
    x1, y1, I1, x2, y2, I2 = vars
    dx1dt = x1**2 * (1 - x1) - y1 + I1
    dy1dt = A * np.exp(alpha * x1) - gamma * y1
    dI1dt = epsilon*(1/60*(1+np.tanh((0.05-x1)/0.001))) - I1
    dx2dt = x2**2 * (1 - x2) - y2 + I2 + theta*(vs-x2)/(1+np.exp(-lamb*(x1-q)))
    dy2dt = A * np.exp(alpha * x2) - gamma * y2
    dI2dt = epsilon*(1/60*(1+np.tanh((0.05-x2)/0.001))) - I2

    return [dx1dt, dy1dt, dI1dt, dx2dt, dy2dt, dI2dt]
```

```
def system(t, vars):
    x1, y1, I1, x2, y2, I2, p = vars
    dx1dt = x1**2 * (1 - x1) - y1 + I1 + theta*rho(p)*(x2 - x1)
    dy1dt = A * np.exp(alpha * x1) - gamma * y1
    dI1dt = epsilon*(1/60*(1+np.tanh((0.05-x1)/0.001))) - I1
    dx2dt = x2**2 * (1 - x2) - y2 + I2 + theta*rho(p)*(x1 - x2)
    dy2dt = A * np.exp(alpha * x2) - gamma * y2
    dI2dt = epsilon*(1/60*(1+np.tanh((0.05-x2)/0.001))) - I2
    dpdt = theta*(x1-x2)

    return [dx1dt, dy1dt, dI1dt, dx2dt, dy2dt, dI2dt, dpdt]
```



Time series tools

1. **Hurst exponent (H)**: measuring persistence.
2. **Sample entropy (SE)**: measuring complexity.
3. **0-1 test (K)**: measuring chaos.
4. **Cross-correlation function (Γ)**: measuring synchrony between neurons.
5. **Kuramoto order parameter (B)**: measuring synchrony between neurons.

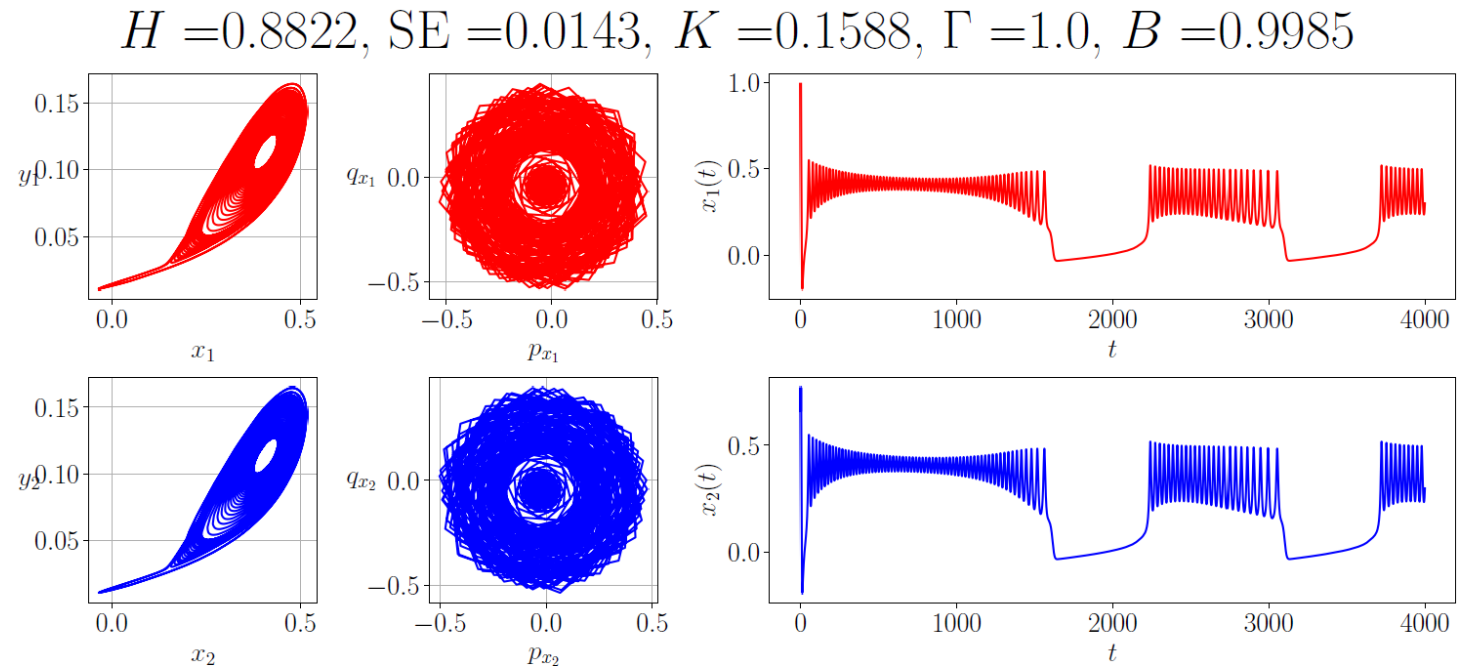


Fig. Applying various tools on the time series generated from simulating the models of coupled neurons.



Hurst exponent (Hurst, 1951)

1. Measures the long-term memory/persistence in time series.
2. Computed using rescaled-range analysis (Qian and Rasheed, 2004).

$$\mathbb{E} \left[\frac{R(t)}{S(t)} \right] = ct^H, \quad t \rightarrow \infty$$

3. $H \in [0,1]$.
4. $H \in [0,0.5)$: anti-persistence (negative dependence on previous values), $H \approx 0.5$: random walk, $H \in (0.5,1]$: positive dependence on previous values.



Sample entropy (Richman & Moorman, 2000)

1. Assesses the complexity of time series data.
2. It is the negative natural log of the probability that if two sets of simultaneous data points of length 'p' have distance less than ' ε ', then the similar thing happens to two sets of simultaneous data points of length 'p+1'.

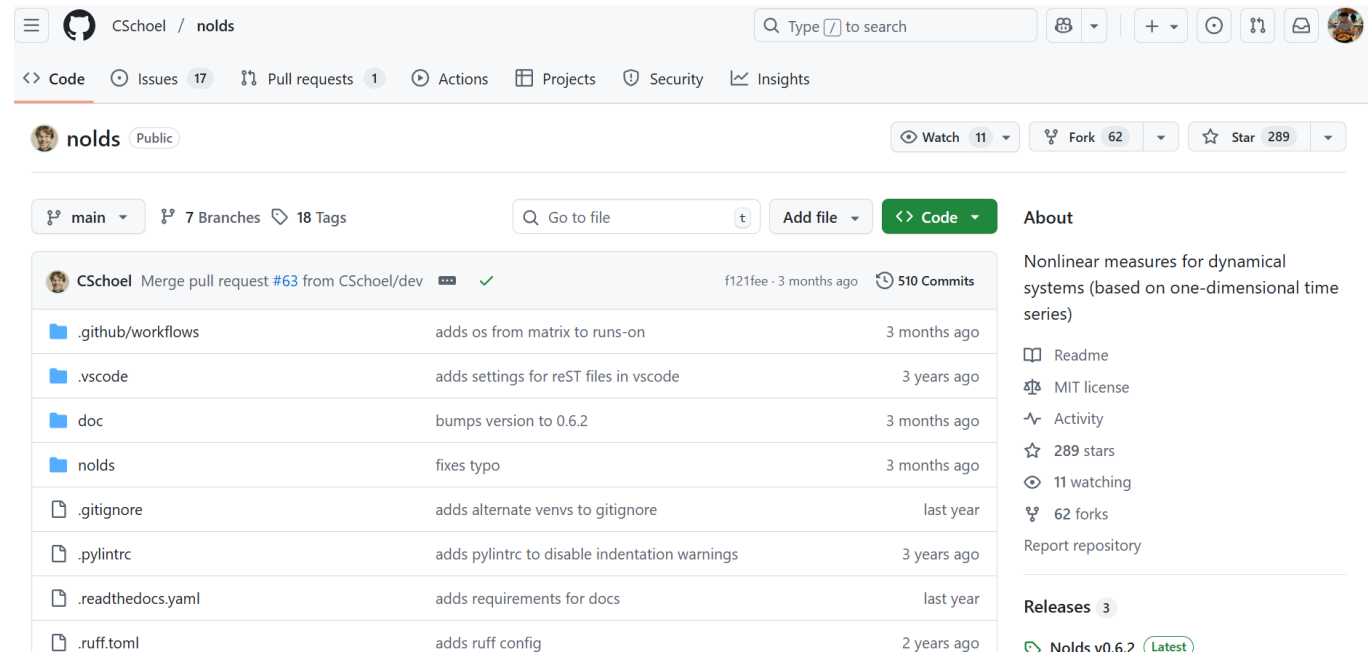
$$SE(p, \varepsilon, N) = \lim_{N \rightarrow \infty} \left(-\log_e \frac{A^p(\varepsilon)}{B^p(\varepsilon)} \right)$$

3. A higher SE indicates higher complexity.
4. Can be normalised between 0 and 1.



‘nolds’ package (Scholzel, 2019)

1. Stands for ‘NOnLinear measures for Dynamical Systems’, based on .
2. Provides functions for directly implementing the rescaled-range based Hurst exponent and also sample entropy to time series.
3. Functions are ``nolds.hurst_rs()`` and ``nolds.sampen()``.
4. Also provides other sophisticated tools for nonlinear measures.



The screenshot shows the GitHub repository for 'nolds' by CSchoel. The repository is public and has 11 watchers, 62 forks, and 289 stars. It contains 7 branches and 18 tags. The repository is described as 'Nonlinear measures for dynamical systems (based on one-dimensional time series)'. The repository includes a README, MIT license, and 3 releases. The latest release is 'Nolds v0.6.2' (Latest). The repository also has 510 commits and a pull request #63 merged from CSchoel/dev.

File	Commit Message	Commit Date
.github/workflows	adds os from matrix to runs-on	3 months ago
.vscode	adds settings for reST files in vscode	3 years ago
doc	bumps version to 0.6.2	3 months ago
nolds	fixes typo	3 months ago
.gitignore	adds alternate venvs to gitignore	last year
.pylintrc	adds pylintrc to disable indentation warnings	3 years ago
.readthedocs.yaml	adds requirements for docs	last year
.ruff.toml	adds ruff config	2 years ago



0-1 test (Gottwald and Melbourne, 2009, 2016)

1. Compute two translated variables from the time series.

$$\tilde{p}(t; e) = \sum_{k=1}^t x(k) \cos(ek),$$

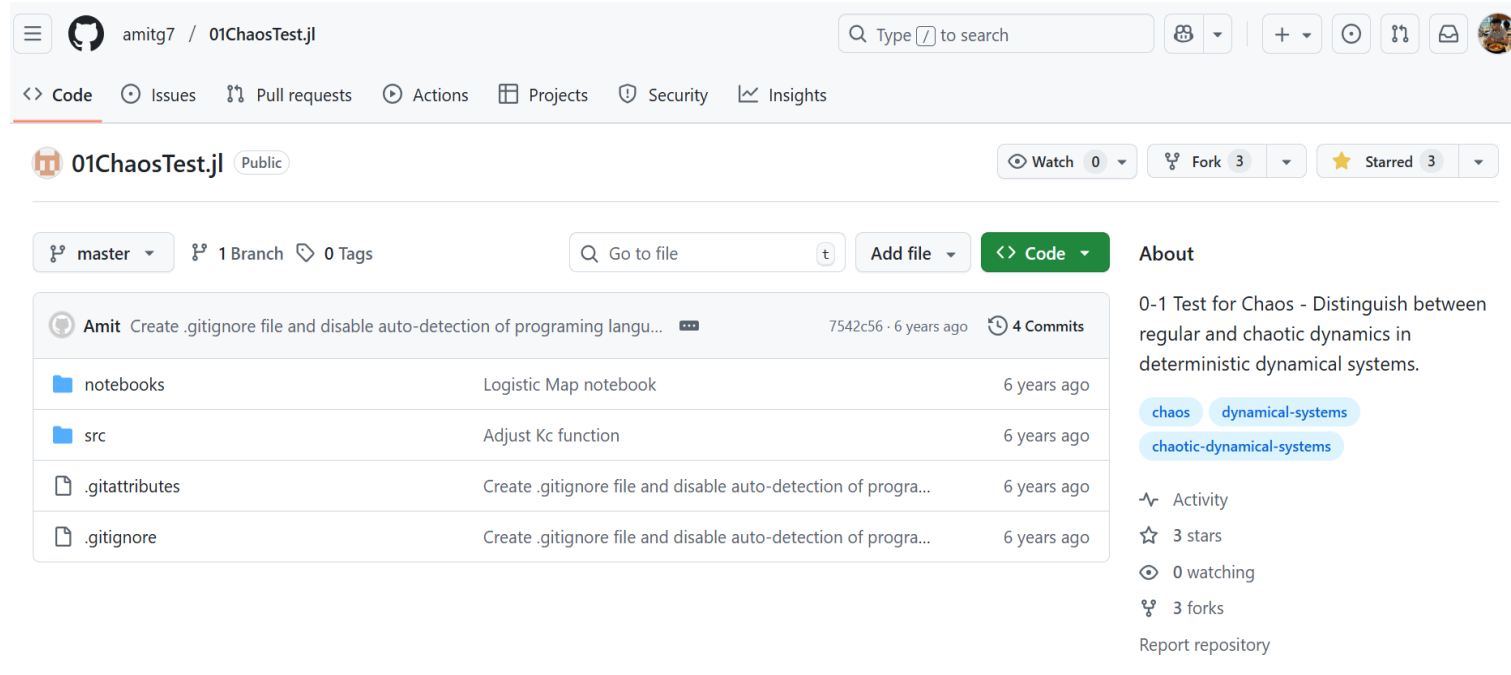
$$\tilde{q}(t; e) = \sum_{k=1}^t x(k) \sin(ek).$$

2. Then compute a mean square displacement term ‘m(t; e)’ and a correction term.
3. Compute the growth rate ‘K’ that quantifies ‘chaos’.
4. ‘K $\approx$ 0’ indicates regular dynamics and ‘K $\approx$ 1’ indicates chaos.



A **julia** package for 0-1 test

1. Translated to .
2. Used ``curve_fit()`` and ``fsolve()`` functions from the ``scipy.optimize`` suite.
3. Used ``pearsonr ()`` function to compute Pearson's correlation coefficient from ``scipy.stats`` suite in one of the steps.





Cross-correlation coefficient (many authors)

1. For two time series from nodes 'i' and 'm', Γ is given by

$$\Gamma_{i,m} = \frac{\langle \tilde{x}_i(n) \tilde{x}_m(n) \rangle}{\sqrt{\langle (\tilde{x}_i(n))^2 \rangle \langle (\tilde{x}_m(n))^2 \rangle}}$$

2. The average is calculated over time and the variation from the mean is

$$\tilde{x}(n) = x(n) - \langle x(n) \rangle$$

3. $|\Gamma| = 1$ indicates total synchrony and $|\Gamma| < 1$ is asynchrony. Moreover, $\Gamma = 1$ represents in-phase synchrony and $\Gamma = -1$ represents anti-phase synchrony.

```
## cross-correlation coeff
phi_x1 = np.array(x1_sol[5000:])
phi_x2 = np.array(x2_sol[5000:])

x1_tilde = phi_x1 - np.mean(phi_x1)
x2_tilde = phi_x2 - np.mean(phi_x2)

Numerator = np.mean(x1_tilde*x2_tilde)
Denominator = np.sqrt(np.mean(x1_tilde**2)*np.mean(x2_tilde**2))

cc = Numerator/Denominator
```



Kuramoto's order parameter (Kuramoto and Battogtokh, 2002)

1. Phase of a neuron 'm' is

$$\zeta_m = \tan^{-1} \left(\frac{y_m(t)}{x_m(t)} \right)$$

2. The complex valued Kuramoto index B is then

$$B_m(t) = \exp(i\zeta_m(t)), \quad i = \sqrt{-1}.$$

3. The index at time 't' is then

$$B(t) = \left| \frac{1}{N} \sum_{m=1}^N B_m(t) \right|.$$

4. When $B = 1$, this means the nodes are all fully coherent and their phases are all locked. Any value $B < 1$ represents incoherence.

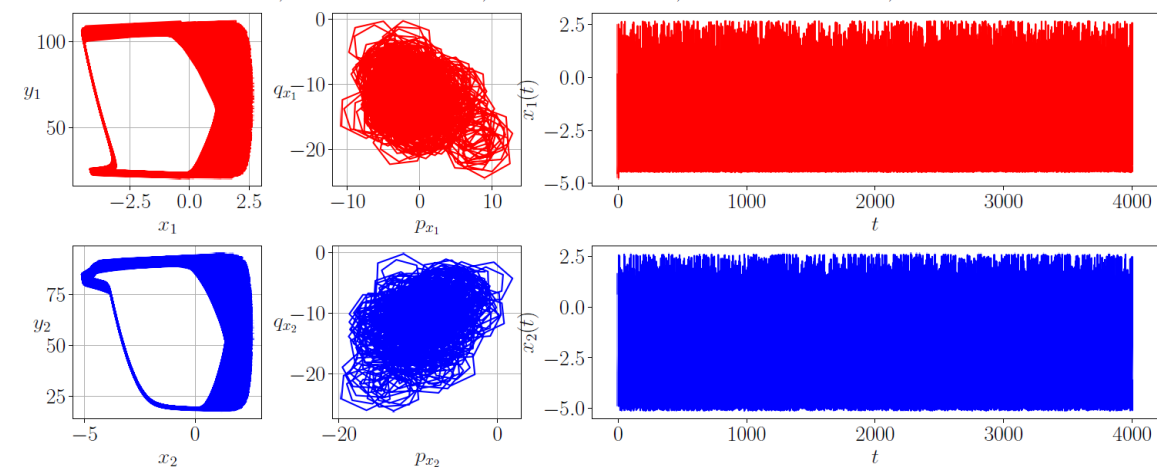
```
## Kuramoto order parameter
l1 = np.arctan(y1_sol/x1_sol)
l2 = np.arctan(y2_sol/x2_sol)

Ind1 = np.exp(1j*l1)
Ind2 = np.exp(1j*l2)
Indt = np.abs(1/2*(Ind1+Ind2))
Kuram = np.mean(Indt)
```

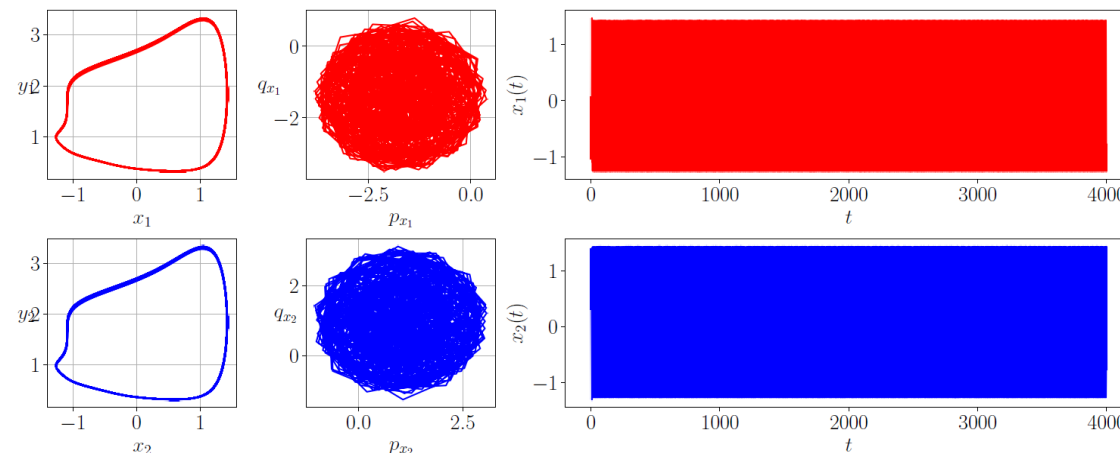


Results from gap-junction coupling

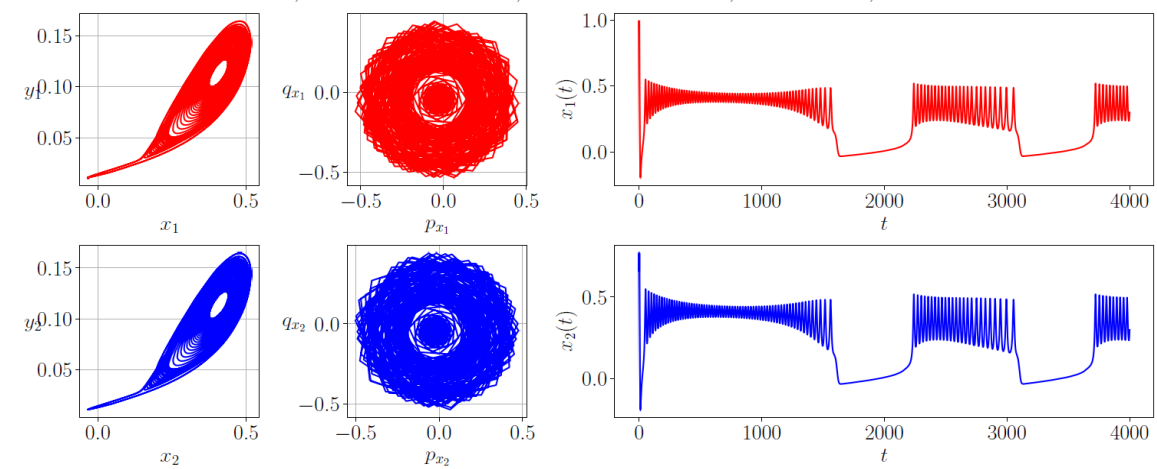
$H = 0.0682$, $SE = 0.049$, $K = 0.973$, $\Gamma = -0.2325$, $B = 0.9448$



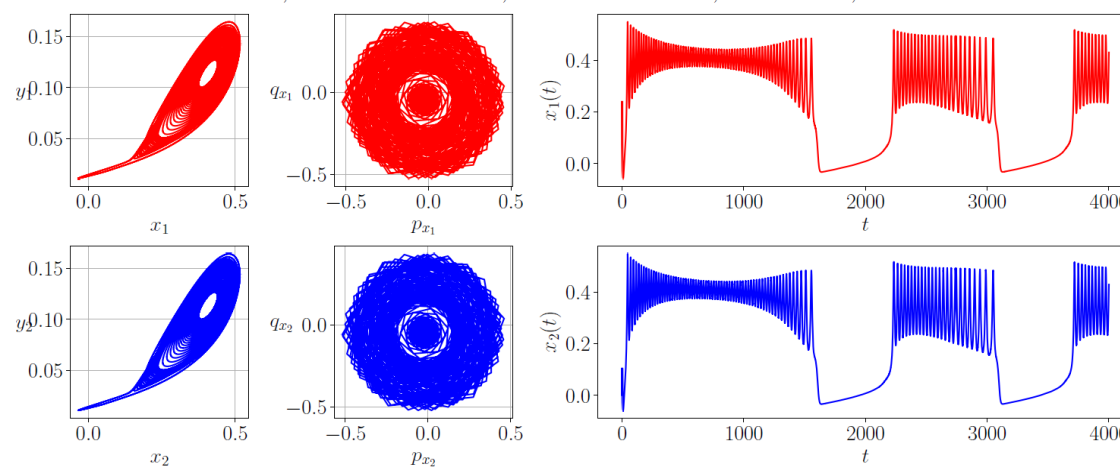
$H = 0.1827$, $SE = 0.0923$, $K = 0.3195$, $\Gamma = -0.7464$, $B = 0.783$



$H = 0.8822$, $SE = 0.0143$, $K = 0.1588$, $\Gamma = 1.0$, $B = 0.9985$



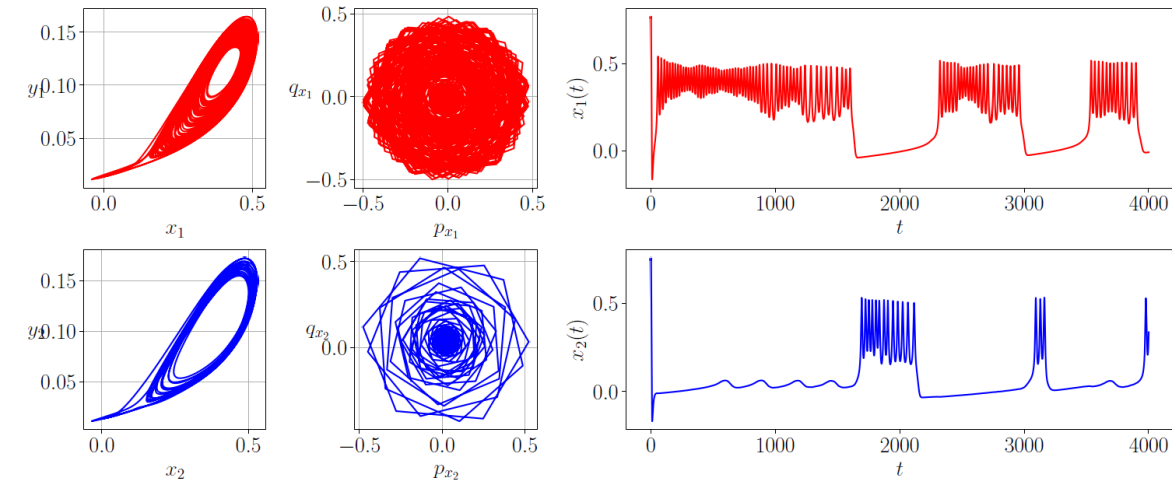
$H = 0.8826$, $SE = 0.0143$, $K = 0.1594$, $\Gamma = 1.0$, $B = 0.9998$



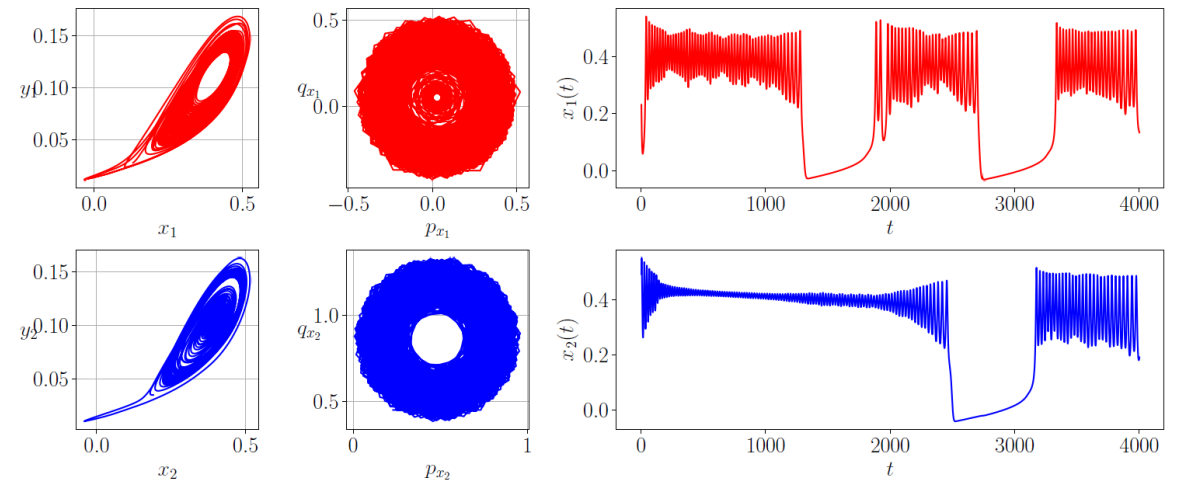


Results from chemical coupling

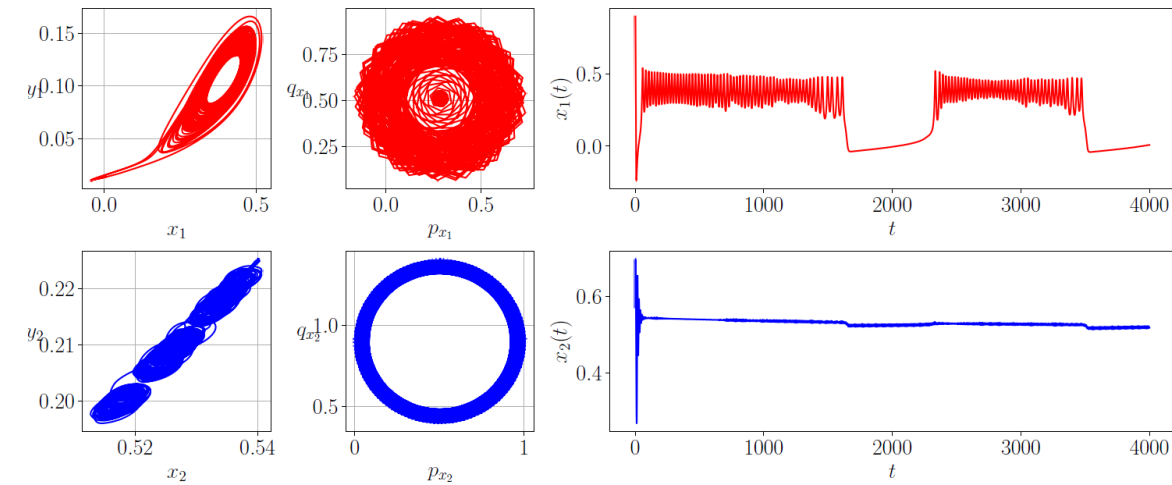
$H = 0.929$, $SE = 0.0077$, $K = 0.0934$, $\Gamma = -0.5153$, $B = 0.9313$



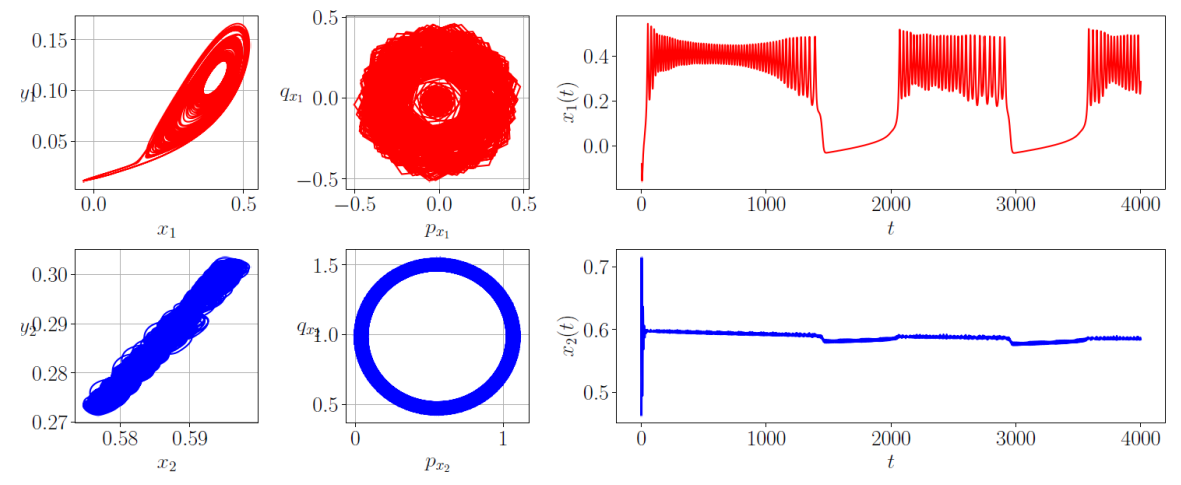
$H = 0.9142$, $SE = 0.0132$, $K = 0.1093$, $\Gamma = 0.3314$, $B = 0.9667$



$H = 0.6073$, $SE = 0.0158$, $K = 0.0803$, $\Gamma = 0.6919$, $B = 0.9687$



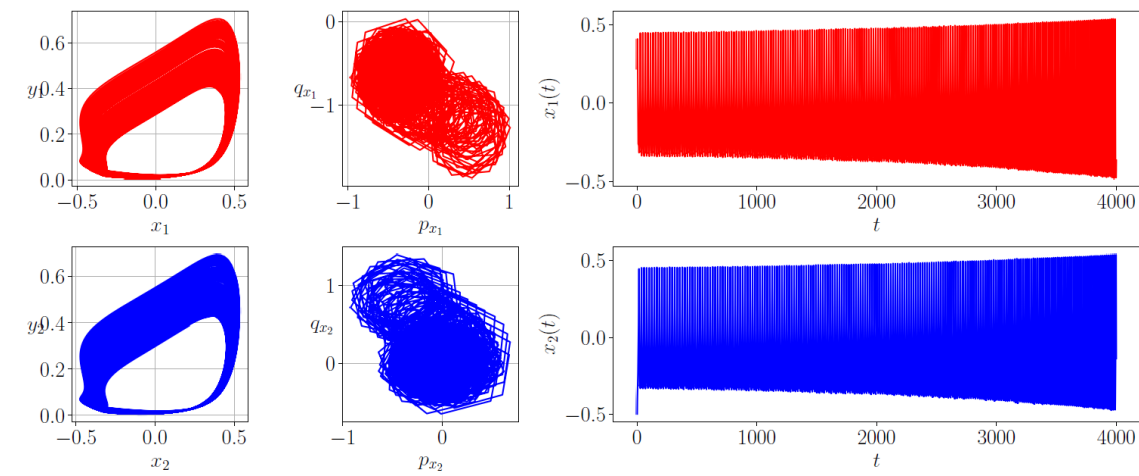
$H = 0.5853$, $SE = 0.027$, $K = 0.0851$, $\Gamma = 0.7929$, $B = 0.9654$



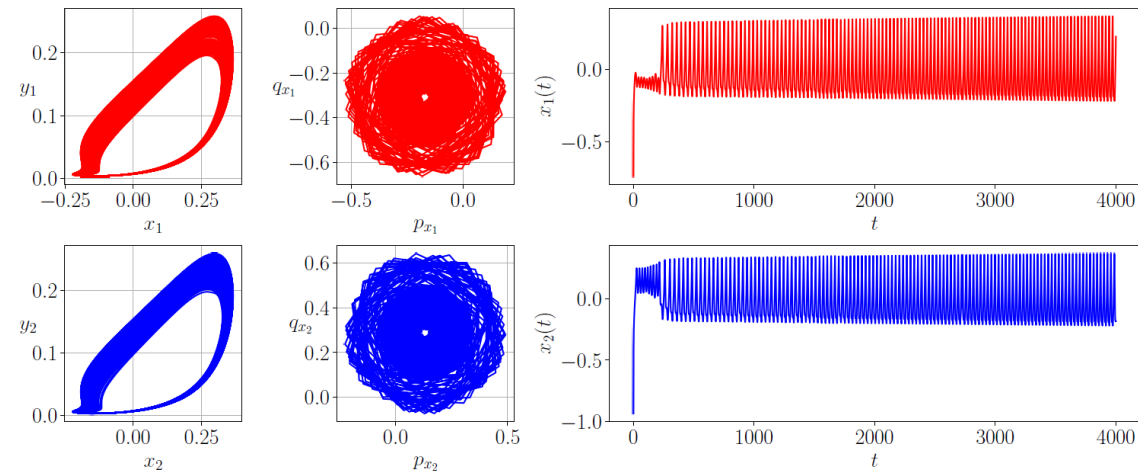


Results from electromagnetic coupling

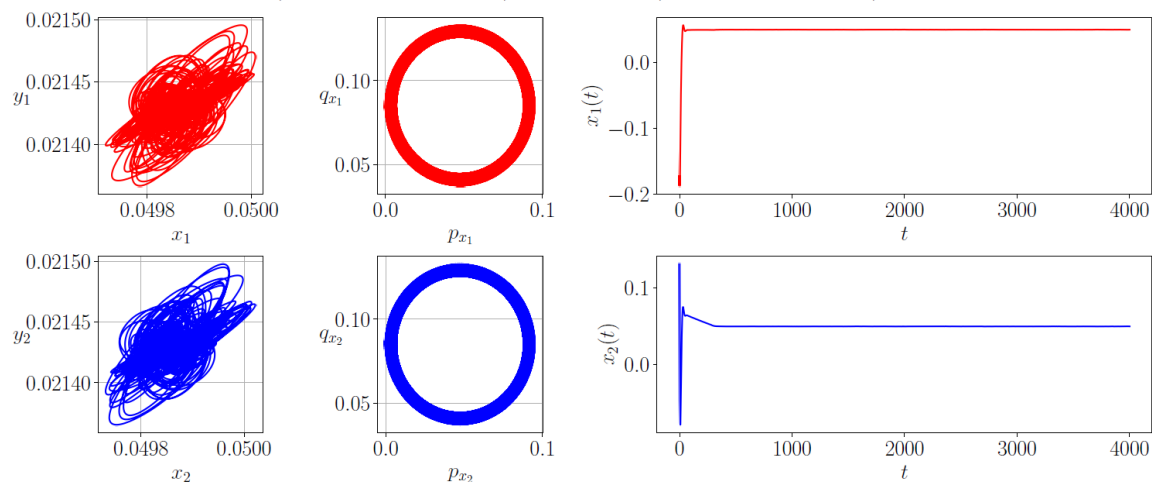
$H = 0.9305$, $SE = 0.0547$, $K = 0.0516$, $\Gamma = -0.713$, $B = 0.9271$



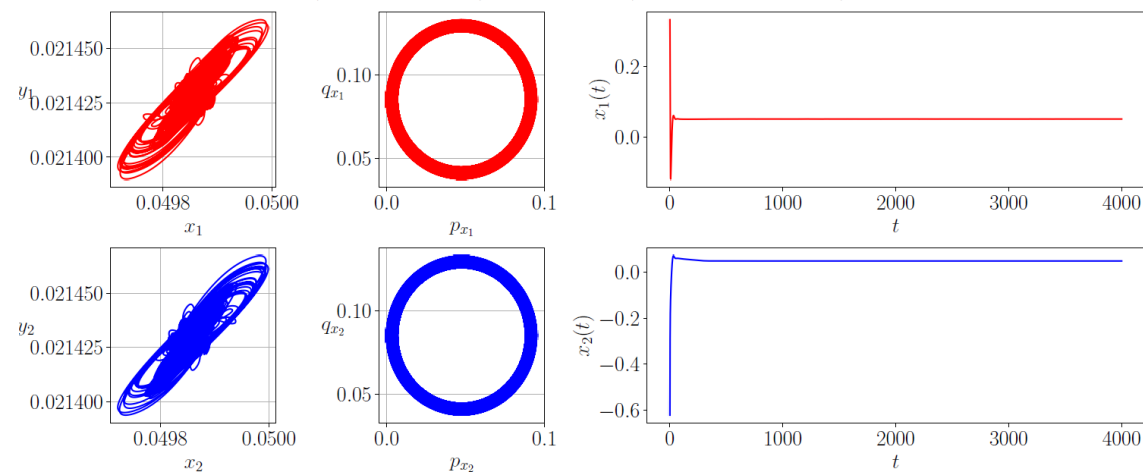
$H = 0.9207$, $SE = 0.0316$, $K = 0.0487$, $\Gamma = -0.7261$, $B = 0.9453$



$H = 0.7652$, $SE = 0.0001$, $K = 0.0$, $\Gamma = -0.9329$, $B = 0.9996$

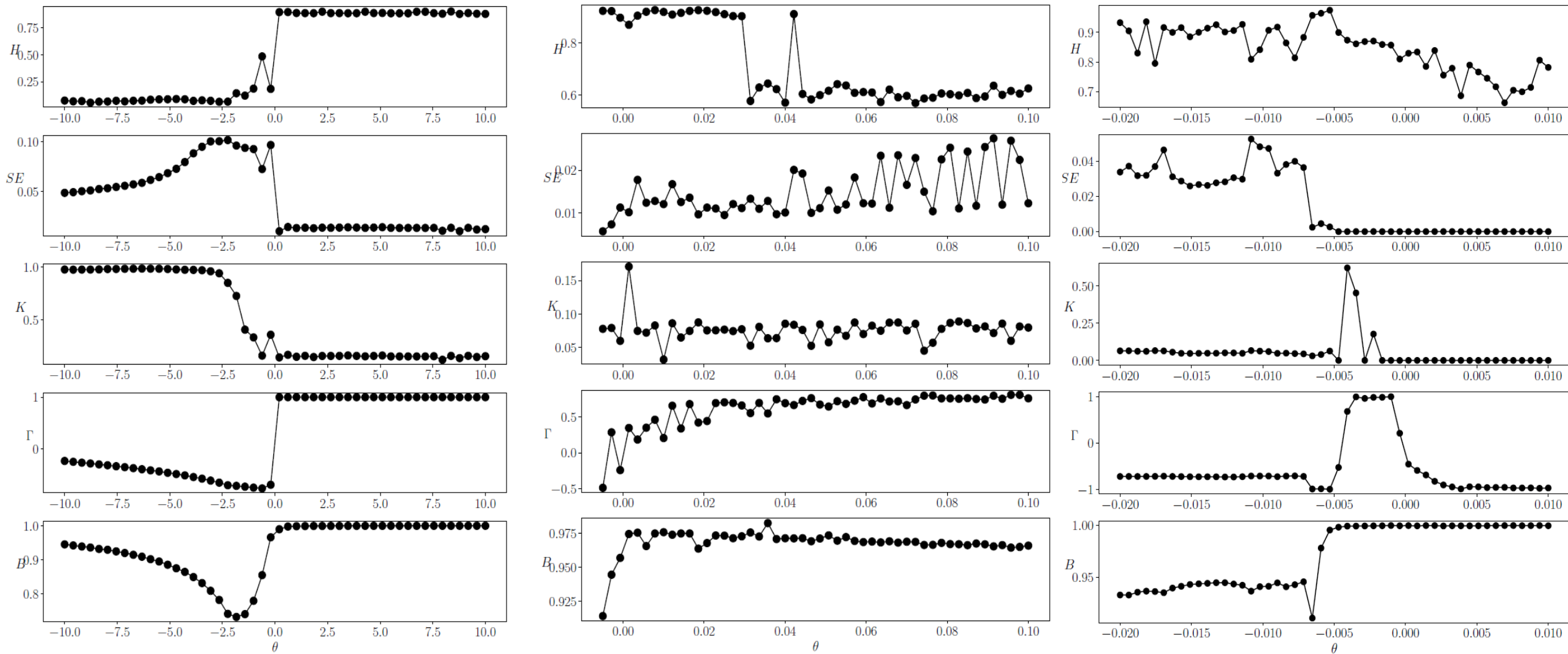


$H = 0.7417$, $SE = 0.0$, $K = 0.0$, $\Gamma = -0.9823$, $B = 0.9996$





Plots with varying coupling strength





Separate experiment from visualization

1. An extra step for data processing before visualization
2. This is where  comes handy! And it's worth it!
3. Read “Taming the Chaos of Computational Experiments” by T. G. Kolda for more on this: siam.org/publications/siam-news/articles/taming-the-chaos-of-computational-experiments/

```
## Code to create the data files

SS = np.linspace(-10, 10, 50)

count = 1
HH=[]
SE=[]
KKTest = []
CC = []
Kuramoto = []
for theta in SS:
    print("count = "+str(count))
    x1_sol, y1_sol, x2_sol, y2_sol, tt, cc, KK1, KK2, Kuram, h1, h2, se1, se2 = bif_gap(theta)
    HH+=(h1+h2)/2, ]
    SE+=(se1+se2)/2, ]
    CC+=cc, ]
    KKTest+=(KK1+KK2)/2, ]
    Kuramoto+=Kuram, ]

    print("H=", (h1+h2)/2)
    print("SE=", (se1+se2)/2)
    print(" ")

    count+=1

df = pd.DataFrame({
    'theta': SS,
    'H': HH,
    'SE': SE,
    'CC': CC,
    'KK': KKTest,
    'Kuramoto': Kuramoto
})

## Create the data file
df.to_csv('data_gap.csv', index=False)
```



Visualization using



In [6]:

```
## Load the data file
dfGap = pd.read_csv('data_gap.csv')
dfGap
```

Out[6]:

	theta	H	SE	CC	KK	Kuramoto
0	-10.000000	0.075755	0.048965	-0.230950	0.974990	0.945371
1	-9.591837	0.068889	0.049600	-0.247435	0.973426	0.942219
2	-9.183673	0.070847	0.050584	-0.264226	0.974289	0.939167
3	-8.775510	0.056358	0.051361	-0.279553	0.974688	0.936165
4	-8.367347	0.065453	0.052763	-0.297540	0.976047	0.931959
5	-7.959184	0.067204	0.053522	-0.314207	0.979496	0.929517
6	-7.551020	0.073805	0.054817	-0.332759	0.980077	0.924822
7	-7.142857	0.068470	0.056010	-0.350708	0.981813	0.920045
8	-6.734694	0.074268	0.057403	-0.369906	0.982537	0.914924
9	-6.326531	0.075233	0.058941	-0.391002	0.983551	0.909313
10	-5.918367	0.083993	0.061830	-0.412140	0.982661	0.901997

```
sz=18
%matplotlib notebook
matplotlib.rc('xtick', labelsz=sz)
matplotlib.rc('ytick', labelsz=sz)

SS = np.linspace(-10, 10, 50)

fig, axs = plt.subplots(5,1, figsize=(10, 12))

axs[0].set_ylabel('$H$', rotation=False, fontsize=sz)
axs[1].set_ylabel('$SE$', rotation=False, fontsize=sz)
axs[2].set_ylabel('$KK$', rotation=False, fontsize=sz)
axs[3].set_ylabel('$\\Gamma$', rotation=False, fontsize=sz)
axs[4].set_ylabel('$B$', rotation=False, fontsize=sz)
axs[4].set_xlabel('$\\theta$', fontsize=sz)

HH = dfGap['H']
SE = dfGap['SE']
KKTest = dfGap['KK']
CC = dfGap['CC']
Kuramoto = dfGap['Kuramoto']

axs[0].plot(SS, HH, 'ko-', ms=10)
axs[1].plot(SS, SE, 'ko-', ms=10)
axs[2].plot(SS, KKTest, 'ko-', ms=10)
axs[3].plot(SS, CC, 'ko-', ms=10)
axs[4].plot(SS, Kuramoto, 'ko-', ms=10)

plt.tight_layout()
```



Summary

1. Coupling induces 'chaos' in the inhibitory regime.
2. Excitatory coupling and its more positive values drive the coupled system into exhibiting bursting. Also, both neurons synchronize.
3. In electromagnetic coupling, excitatory coupling drives the system to decay oscillation, falling into a symmetric equilibrium point.
4. Future work: move to GPU accelerated framework using `CuPy`:
5. Future work: use a complex network of neuron and integrate `NetworkX`:





Acknowledgements



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Reference: Indranil Ghosh, Hammed Olawale Fatoyinbo, and Sishu Shankar Muni, “Time series analysis of coupled slow-fast neuron models: From Hurst exponent to Granger causality” (2025), arxiv.org/abs/2507.13570.



repository: github.com/indrag49/TS-SlowFast-dML